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NYO-7968

SHERWOOD PROGRESS REPORT

NO. 1

by

Harold Grad

October 1, 1957

institute of mathematical sciences

NEW YORK UNIVERSITY

NEW YORK, NEW YORK

NYO-7968
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Controlled Thermonuclear Processes (M-3679, 20th Ed.)
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Contract No. AT(30-1)-1480

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PREFACE

This report is a summary of activities carried on at the Institute of Mathematical Sciences at New York University from March 1954 when NYU first entered the Sherwood program until June 1957. It is intended to be the first in a series of semi-annual reports.

August 9, 1957

SURVEY

The scientific staff working on Sherwood at NYU at the present time (July, 1957) consists of a theoretical group of 19 individuals equivalent to 15 full-time persons and an experimental group of 4 individuals amounting to 3 persons full-time. This is a growth from an average of 4 full-time persons in 1954 to 7 in 1955, and 12 in 1956. The present total of 23 individuals (18 full-time persons) breaks down into 11 mathematicians (8 full-time), 5 physicists and chemists (3 full-time), and 7 assistants, technicians, and computers (all full-time).

The subjects and methods of investigation have been strongly influenced by the preponderance of mathematicians and an emphasis on general theory derived from the absence of a focus on any specific experimental device. In particular, a coordinated effort has been made to develop the basic mathematical theory of plasma physics and magnetohydrodynamics with an eye to its ultimate use in the quantitative analysis of thermonuclear plasmas; however the study of specific problems of a possibly more transient but more immediate significance has not been neglected. Some of the specialties of the IMS which have been brought to bear on Sherwood problems are the theory of nonlinear partial differential equations (including gas dynamics), variational analysis, asymptotic theory of ordinary differential equations, numerical analysis, electro-

magnetics and kinetic theory. It is expected that increasing use will be made of the computational aid offered by the UNIVAC and recently acquired IBM 704.

Contributions have been made and are under way in the general theory of fluid magnetics, especially in the fields of nonlinear wave motion and equilibrium and in questions of stability (linear and nonlinear); to the theory of particle motion in electromagnetic fields, including adiabatic and non-adiabatic theories and also self-consistent field problems; to the theory of the Boltzmann equation including the derivation of generalized fluid models, the theory of shock waves, and evaluation of transport coefficients; also to questions of particle losses, in particular for cusped geometries.

An early aim, partly motivated by the original Sherwood emphasis on particle models and partly by the mathematical skills available at the IMS, was to recognize and, wherever necessary, discover mathematical structures analogous to those of classical fluid dynamics. Throughout, an attempt has been made to combine the complementary features of particle and fluid analysis. This interrelation is perhaps the most distinctive feature of a thermonuclear plasma. Only in certain very special cases can complete self-consistent particle solutions be obtained. On the other hand, although self-consistency is automatic in a continuum treatment, there may be a significant loss of detail. The best procedure has been found

to be a preliminary fluid analysis followed by a particle improvement.

This summary report is presented chronologically for the years 1954, 1955, 1956, and the first half of 1957. A list of references is given at the end.

1954

The major subject of investigation was "classical" hydromagnetics based on one of the forms of the equations proposed by Lundquist* for a single conducting fluid

*S. Lundquist. Studies in Magnetohydrodynamics, Arkiv for Fysik, 5 (1952).

described by a scalar pressure and satisfying Ohm's law and Maxwell's equations.

Mathematical Analogues

A very useful tool early in the development of hydromagnetic theory was the discovery of certain exact mathematical analogues in some special cases between the hydromagnetic equations and the conventional equations of fluid and gas dynamics. The importance of such an observation is the possibility of immediately transferring to hydromagnetic theory a whole block of previously developed theory.

A large class of two-dimensional problems was found to exhibit an exact mathematical analogy to certain ordinary (non-magnetic) fluid problems if the "total" pressure, $p^* = p + B^2/2\mu$, and "total" energy per unit mass $e^* = e + B^2/2\mu\rho$ are introduced as variables (references [1] and [7a]). All of the tools available for ordinary gas dynamics are applicable to these problems, for example to solve many two-dimensional

problems involving nonlinear wave motion, nonuniform shock motion, and the interaction of waves. Specifically, this analogue is observed to hold in a geometry in which the fluid flow is two-dimensional (x and y) while the magnetic field has only a z-component which, however, can vary arbitrarily in x and y (also t).

The equilibrium problem for a perfect conductor, viz. solution of the system

$$\nabla p = \frac{1}{\mu}(\text{curl } B) \times B$$

$$\text{div } B = 0$$

was observed to be an exact mathematical analogue of ordinary incompressible, rotational flow after making the identifications $B \sim$ fluid velocity, $-p \sim$ Bernoulli constant (references [1] and [7a]). Later developments in this subject have brought this theory to a state beyond what was previously known for fluid flow (report to appear, November 1957). This problem of finding equilibrium configurations was shown to be equivalent to the variational problem of extremizing $\int (B^2/2\mu - p)dV$ or merely $\int (B^2/2\mu)dV$ subject to a variety of side conditions and constraints (reference [7a]).

Stability

For the special case of a pure plasma (no magnetic field) separated by a sharp interface (sheet current) from a vacuum

magnetic field (no plasma), an energy principle was formulated for the investigation of the stability of such equilibria (Princeton, October 1954 [7a]). Specifically, it was found that a minimum of $\int(B^2/2\mu)dv$ (or of some modified functional, depending on the constraints) would imply stability rather than mere equilibrium. The relation between stability and maximum inductance was brought out. It was also found that for a geometry similar to that of the cylindrical pinch, the effects of the plasma and of the magnetic field on stability could be separated (Princeton, October 1954 [7b]). This theory has since been superseded by complete quantitative analyses.

Electromagnetic Theory

As a tool in obtaining these stability results, it was found necessary to first develop the theory of Maxwell's equations without displacement current (the properties of which are surprisingly different from those of Maxwell's equations (reference [7a] also [2-V]). It is in principle preferable to use a Galilean invariant system (i.e. Maxwell's equations, but, more important, the theory of this modified system is much simpler and allows more difficult problems to be treated.

General Hydromagnetic Theory

The general nonequilibrium and nonlinear theory of the classical hydromagnetic equations was put on a firm foundation by the discovery that the set of differential equations is

symmetric hyperbolic in the case of infinite conductivity. Explicit solutions for certain "simple waves" provide tools for the solution of a variety of problems (references [4] and [7b]).

It was found that the relativistic equations of motion have the same type of mathematical structure as the classical magnetohydrodynamic equations and can be as easily treated. Summarizing, both the classical and relativistic hydromagnetic equations were put on as firm a foundation as the equations of classical gas dynamics.

1955

Stability

The energy principle for the stability of free surfaces which had been previously formulated (Princeton, October 1954 [7a]) was found to lead to necessary and sufficient conditions for linear (small displacement) stability in terms of the magnetic field gradient at the interface or, equivalently, the local curvature of magnetic lines on the interface (Berkeley, February 1955 [8c]). This result was proved for constant flux constraints and for constant current constraints as well as for a fluid (plasma) with any equation of state. The major mathematical tool was variational analysis based on the "pre-Maxwell" equations (reference [2-V]).

The stability theory was extended to allow the explicit comparison of the effects of acceleration and magnetic line curvature on stability in arbitrary geometries. Quantitatively, an acceleration of magnitude α has the same effect on stability as a magnetic line curvature $1/R = \alpha/A^2$, where $A = B^2/\mu\rho$ is the Alfven speed. In sign, a compressive acceleration of the plasma is destabilizing, and an accelerated expansion is stabilizing. For example, a rebounding pinch (expanding outward) would be stable if accelerating fast enough. Also, the criterion for stability with respect to finite displacements was derived (reference [9a]). A special

case of "dynamical" stability was solved namely, Taylor instability with a rotating magnetic field (reference [9b]). The result obtained was that, although rotation of the magnetic field reduces instability (in particular, growth rates remain bounded), it does not eliminate it. The unique feature of the mathematically thin interface was brought out by the proof that, except in the case of sharp interface geometries, every sufficiently localized disturbance is stable; in other words instabilities can arise only through cooperative behavior involving a finite part of the system (reference [9c]). The reference length is the distance over which the magnetic field changes appreciably; only in the presence of an infinite sheet current can an arbitrarily small scale disturbance be unstable.

Cusped Geometries

A number of absolutely stable cusped configurations were obtained from this stability theory (reference [8c] also [5-I]). The modification of this fluid theory to include fine-grain particle effects led to an estimate of particle losses through the cusps (LASL, June 1955 - no report issued). Numerical estimates of these losses in regions of thermonuclear interest (Princeton, October 1955 [9]) indicated that cusped geometries were possibly competitive with others. It was shown that the efficacy of this geometry depended crucially

on obtaining the thinnest possible boundary layer; arguments were given to indicate the theoretical possibility of attaining a thickness as small as the electron gyro radius (reference [9d], for more recent results see also [10b] and [5-I]). Part of the problem involved obtaining a self-consistent particle analysis of the boundary layer.

Generalized Hydromagnetics

A proposed modification of the classical hydromagnetic equations was developed (the Guiding-Center Fluid, Princeton, October 1955 [9h]). The essential feature, macroscopically, is the introduction of a possibly strongly anisotropic stress tensor induced by the magnetic field. Specifically, this incorporates the mirror effect (which does not exist in the classical equations) into a macroscopic theory. A greatly simplified macroscopic derivation of these equations in terms of an appropriate entropy was later developed (Los Alamos, Theoretical Seminar, December 1955 - no report issued). Incidental to the development of these equations, it was found to be necessary to obtain a more exact description of the guiding-center motion of a single particle than was previously available (reference [9h]).

Miscellaneous

Various questions were treated relating to diffusion across magnetic fields, (reference [8b] also [3]); the validity of Ohm's law for a plasma including certain superconductive

aspects, (reference [8d]); structure of a one-fluid hydromagnetic shock, (reference 9f]). An exact solution was found for the motion of a particle in a spatially homogeneous but time varying field and the result was applied to the computation of nonadiabatic heating (magnetic pumping) produced by a field oscillating in the vicinity of the gyro frequency, (reference [9g]). Further results were obtained in the variational theory of equilibrium solutions of $\nabla p = J \times B$. A program of applying the method of moments to the Boltzmann equation with magnetic field was initiated.

Experimental

A small experimental program was started in 1955. The goal was to design experiments which would verify some aspect of theory rather than to specifically advance the art of producing thermonuclear plasmas. The initial investigation was of plasma guns of various types. It was found that the essential features were not dependent on the specific geometry, and it was verified that substantial diamagnetic plasmas were formed and accelerated (references [9e] and [6]).

1956

Particle Motion

With regard to the motion of particles in an electromagnetic field, several new adiabatic invariants generalizing the conventional magnetic moment were found. By the use of one of these, it was found that the criterion for containment in a mirror machine was only that the gyro radius be small compared to the longitudinal variation of the field; there is no nonadiabatic effect when the field changes appreciably in the radial direction in a single gyro radius (UCRL, February 1956 - no report issued, also [10f]). A general analysis of the equations of motion revealed the existence of several distinct types of guiding-center-like motions rather than just one (reference 10f]).

Particle Losses

The distinction between particle losses through a mirror and through a cusp was clarified by consideration of the generalized adiabatic invariants just mentioned and by a generalization of the concept of "mirror ratio" to make it applicable to a cusp (where the magnitude of B does not increase as one approaches the cusp) as well as to a mirror (reference [10b]). The entirely different theories of particle losses for cusps and for mirrors turns out to be due to a distinction between large and small effective mirror ratios

rather than one basic to the geometries themselves. For example, a combination of both types of argument is needed to compute particle losses in a general picket fence geometry with no sharp boundary.

The analysis of cusped geometries was pursued in several directions. Methods of computing their shapes in both two and three dimensions were developed (references [10d] and [10e]). The effect of charge separation on losses was estimated (reference [10b]). The insertion of an axial (crossed) field was suggested to cut particle losses (Figure 1a) and the insertion of an internal conductor carrying current to improve the economics was proposed (Figure 1b) (LASL, October 1956 - no report issued). Also, the possibility was mentioned of producing an <u>end pinch</u> to reduce end losses in a linear cusped pinch geometry.

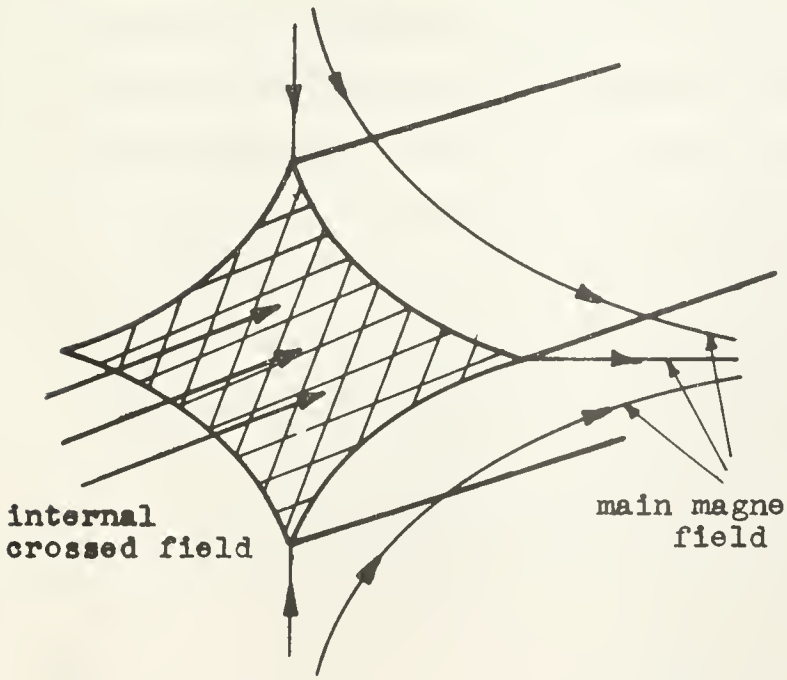

Figure 1a

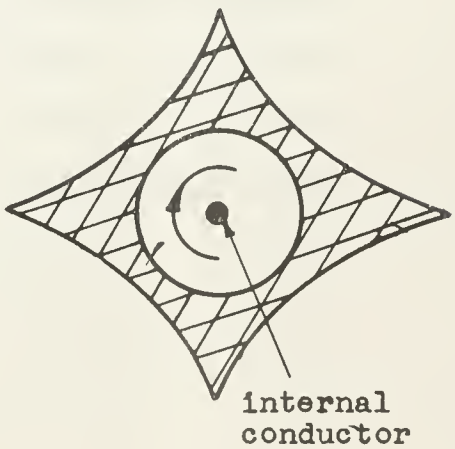

Figure 1b

Stability

The effect on stability of certain types of external lumped circuits was investigated (reference [10c]). General mathematical methods were developed of estimating the stabilizing effect of crossed fields and external conductors in arbitrary geometries which do not permit separation-of-variables techniques.

Shock Waves

A program investigating the efficacy of shock heating was initiated. It was found that ion heating can be dominant under certain easily attainable conditions (UCRL, February 1956 - no report issued). The principal assumption to be verified is that something which can be called a shock exists, and most of the subsequent effort has been devoted to this question. It was found that the Chapman-Enskog treatment of the Boltzmann equation was completely inadequate to treat this problem, and new mehtods were found to deal with it. A sequence of fluid-like approximations to the Boltzmann equation was set up. The first of these, equivalent to the guiding-center fluid (see above, 1955), yielded no shock transition but suggested that the solution, when found, would be oscillating. Preliminary results based on a second approximation (Gatlinburg, June, 1956 [10g] were later also found to be short of the mark, but gave other interesting information.

Miscellaneous

Investigations related to Ohm's law were continued and, in particular it was found that the approximate relation $E + u \times B = 0$ is set up by two entirely distinct mechanisms, viz. collisions on the one hand and plasma or gyro oscillations on the other, in two entirely disconnected ranges of physical parameters. The theory of the guiding-center fluid equations was extended slightly, and the basic mathematical similarity to the classical equations was demonstrated.

Experimental

In view of the preliminary positive results of the plasma gun study, facilities for the study of the super-sonic motion of plasma streams in magnetic fields were further developed. Some further properties of plasmas from plasma guns were measured, and a study was initiated of the behavior of these high-speed plasmas in various magnetic field configurations. It was also proposed to investigate a number of theoretically predicted stable configurations by means of liquid metal analogue experiments, and a feasibility study was completed.

January to June 1957

Stability

The investigation of stable configurations was continued, and a few new ones were found (reference [11c]). Some tentative results were obtained on the effect of conducting end-plates on increasing the maximum stable value of β in a mirror machine (reference [11b]). This work is being continued. The curvature criterion for stability of the sharp interface was shown to be unaltered if the plasma was taken to be subject to the Boltzmann equation (rather than to satisfy conventional fluid equations). The same is true for stability with respect to finite perturbations (reference [12b]).

Cusped Geometries

A computation of cusp losses was made in a special case with crossed fields and the expectation of a drop in loss rate was confirmed (reference [11d]).

Shock Waves

The question of the existence of a shock transition without the intervention of collisions as a mechanism for irreversibility was pursued further. A qualitative particle description indicating such a possibility was described (reference [11f]). The necessary entropy change was seen to be linked with nonadiabaticity of the particles and randomization of phase. From the particle viewpoint, the

self-consistent field problem presents a number of unusual mathematical features, and preparations have been made to attack this problem numerically. From the fluid viewpoint, the "first approximation" mentioned above (1956) was investigated in more detail and the wavelength of the oscillations was found to be such as to confirm the adiabaticity of the electrons and consequent dominance of ion heating (references [11f] and [11g]). The next step is to investigate equations for the third approximation (the second was found to yield no shock transition). In a slightly different direction, the inclusion of collisions as a modification of the first approximation was observed to yield a shock transition (reference [11g]). These results are probably valid for some part of the intermediate region between the one covered by the Chapman-Enskog equations and the extreme case of no collisions. Although the question is not yet definitely answered, the indication is that for most efficient heating of ions, a shock of some intermediate strength is likely to be optimum.

Particle Motion

The formal guiding-center expansion of Kruskal was shown to be a legitimate, asymptotically convergent expansion (reference [11e]). It was found that the various guiding-center-like motions mentioned above (1956) could be simply described in terms of a two-parameter expansion generalizing

the conventional one-parameter guiding-center expansion.

Miscellaneous

A number of developments were made in the theory of Ohm's law, in particular with relation to the "paramagnetic" effect (reference [11a] and in the numerical computation of the electrical conductivity.

Some additional results were obtained in the theory of equilibrium configurations (Princeton, April 1957 - not issued).

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